

# Equivariant mean field flow

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## Abstract

We consider a gradient flow associated to the mean field equation on  $(M, g)$  a compact riemannian surface without boundary. We prove that this flow exists for all time. Moreover, letting  $G$  be a group of isometry acting on  $(M, g)$ , we obtain the convergence of the flow to a solution of the mean field equation under suitable hypothesis on the orbits of points of  $M$  under the action of  $G$ .

*Key words* : Mean field equation, Geometric flow.

**AMS subject classification** : 35B33, 35J20, 53C44, 58E20.

## 1 Introduction

Let  $(M, g)$  be a compact riemannian surface without boundary, we will study an evolution problem associated to the mean field equation :

$$\Delta u + \rho \left( \frac{f e^u}{\int_M f e^u dV} - \frac{1}{|M|} \right) = 0, \quad (1.1)$$

where  $\rho$  is a real parameter,  $|M|$  stands for the volume of  $M$  with respect to the metric  $g$ ,  $f \in C^\infty(M)$  is a given function supposed strictly positive and  $\Delta$  is the Laplacian with respect to the metric  $g$ . The mean field equation appears in statistic mechanic from Onsager's vortex model for turbulent Euler flows. More precisely, in this setting, the solution  $u$  of the mean field equation is the stream function in the infinite vortex limit (see [5]). This equation is also linked to the study of condensate solutions of the abelian Chern-Simons-Higgs model (see for example [4], [12], [20], [22]). Equation (1.1) is also related to conformal geometry. When  $(M, g)$  is the standard sphere and  $\rho = 8\pi$ , the problem to find a solution to equation (1.1) is called the Nirenberg Problem. The geometrical meaning of this problem is that, if  $u$  is a solution of (1.1), the conformal metric  $e^u g$  admits a gaussian curvature equal to  $\frac{\rho f}{2}$ .

Equation (1.1) is the Euler-Lagrange equation of the nonlinear functional

$$I_\rho(u) = \frac{1}{2} \int_M |\nabla u|^2 dV + \frac{\rho}{|M|} \int_M u dV - \rho \log \left( \int_M f e^u dV \right), \quad u \in H^1(M). \quad (1.2)$$

By using the well-known Moser-Trudinger's inequality (see inequality (3.2)), one can easily obtain the existence of solutions of (1.1) for  $\rho < 8\pi$  by minimizing  $I_\rho$ . Existence of solutions becomes much harder when  $\rho \geq 8\pi$ . In fact, in this case, the functional  $I_\rho$  is not coercive. The existence of solutions to equation (1.1) has been intensively studied these last decades when  $\rho \geq 8\pi$ . Many partial existence results have been obtained according to the value of  $\rho$  and to the topology of  $M$  (see for example [6], [8], [14], [19] in the references therein). Recently, Djadli [9] proves the existence of solutions to (1.1) for all riemannian surfaces when  $\rho \neq 8k\pi$ ,  $k \in \mathbb{N}^*$ , by studying the topology of sublevels  $\{I_\rho \leq -C\}$  to achieve a min-max scheme (already introduced in Djadli-Malchioldi [10]).

In this paper, we consider the evolution problem associated to the mean field equation, that is the following equation

$$\begin{cases} \frac{\partial}{\partial t} e^u = \Delta u + \rho \left( \frac{f e^u}{\int_M f e^u dV} - \frac{1}{|M|} \right) \\ u(x, 0) = u_0(x), \end{cases} \quad (1.3)$$

where  $u_0 \in C^{2+\alpha}(M)$ ,  $\alpha \in (0, 1)$ , is the initial data. It is a gradient flow with respect to the following functional :

$$E_f(u) = \frac{1}{2} \int_M |\nabla u|^2 dV + \frac{\rho}{|M|} \int_M u dV - \rho \ln \left( \int_M f e^u dV \right), \quad u \in H^1(M). \quad (1.4)$$

We first prove the global existence of the flow (1.3). We obtain the following result :

**Theorem 1.1.** *For all  $u_0 \in C^{2+\alpha}(M)$  ( $0 < \alpha < 1$ ), all  $\rho \in \mathbb{R}$  and all function  $f \in C^\infty(M)$  strictly positive, there exists a unique global solution  $u \in C_{loc}^{2+\alpha, 1+\frac{\alpha}{2}}(M \times [0, +\infty))$  of (1.3).*

Next, we investigated the convergence of the flow when the initial data and the function  $f$  are invariant under an isometry group acting on  $(M, g)$ . A lot of works has been done for prescribed curvature problems invariant under an isometry group, we refer to [1], [13], [16], [17] and the references therein. Before given a more precise statement of our results, we introduce some notations. Let  $G$  be an isometry group of  $(M, g)$ . For all  $x \in M$ , we define  $O_G(x)$  as the orbit of  $x$  under the action of  $G$ , i.e.

$$O_G(x) = \{y \in M : y \in \sigma(x), \forall \sigma \in G\}.$$

$|O_G(x)|$  will stand for the cardinal of  $O_G(x)$ . We say that a function  $f : M \rightarrow \mathbb{R}$  is  $G$ -invariant if  $f(\sigma(x)) = f(x)$  for all  $x \in M$  and  $\sigma \in G$ . We define  $C_G^\infty(M)$  (resp.  $C_G^{2+\alpha}(M)$ ,  $\alpha \in (0, 1)$ ) as the space of functions  $f \in C^\infty(M)$  (resp.  $f \in C_G^{2+\alpha}(M)$ ) such that  $f$  is  $G$ -invariant. We prove the convergence of the flow under suitable hypothesis on  $G$  allowing us also to handle the critical case when  $\rho = 8k\pi$ ,  $k \in \mathbb{N}^*$ .

**Theorem 1.2.** *Let  $G$  be an isometry group acting on  $(M, g)$  such that*

$$|O_G(x)| > \frac{\rho}{8\pi}, \quad \forall x \in M,$$

*and  $f \in C_G^\infty(M)$  be a strictly positive function. Then, for all initial  $u_0 \in C_G^{2+\alpha}(M)$ , the global solution  $u \in C_{loc}^{2+\alpha, 1+\frac{\alpha}{2}}(M \times [0, +\infty))$  of (1.3) converges in  $H^2(M)$  to a function  $u_\infty \in C_G^\infty(M)$  solution of the mean field equation (1.1).*

Assuming that  $f$  is a positive constant and  $G = \text{Isom}(M, g)$ , the group of all isometry of  $(M, g)$ , we obtain :

**Corollary 1.1.** *Suppose that for all  $x \in M$ , we have  $|O_G(x)| = +\infty$  then, for all  $\rho \in \mathbb{R}$ , the solution of the flow (1.3) converges in  $H^2(M)$  to a function  $u_\infty \in C_G^\infty(M)$  solution of the mean field equation (1.1).*

**Remark 1.1.** *Taking  $M = \mathbb{S}^1 \times \mathbb{S}^1$  endowed with the product metric and  $G = \text{Isom}(M, g)$ , we have, for all  $x \in M$ ,  $|O_G(x)| = +\infty$ .*

If  $f$  isn't constant, we also have :

**Corollary 1.2.** *If  $\rho < 16\pi$  and  $f \in C^\infty(\mathbb{S}^2)$  is an even function then the flow (1.3) converges in  $H^2(\mathbb{S}^2)$  to an even function  $u_\infty \in C^\infty(\mathbb{S}^2)$  solution of the mean field equation.*

The plan of this paper is the following : in Section 2, we will prove Theorem 1.1. In Section 3, we will give an improved Moser-Trudinger inequality for  $G$ -invariant functions. In Section 4, we establish Theorem 1.2 : first, using our improved Moser-Trudinger inequality, we obtain a uniform (in time)  $H^1(M)$  bound for the solution  $u(t)$  of (1.3) where  $u(t) : M \rightarrow \mathbb{R}$  is defined by  $u(t)(x) = u(x, t)$ . Then, from the previous estimate, we will derive a uniform  $H^2(M)$  bound. Theorem 1.2 will follow from this last estimate.

## 2 Proof of Theorem 1.1.

In this section, we prove the global existence of the flow (1.3). We begin by noticing that, since the flow (1.3) is parabolic, standard methods provide short time existence and uniqueness for it. Thus, there exists  $T_1 > 0$  such that  $u \in C^{2+\alpha, 1+\frac{\alpha}{2}}(M \times [0, T_1])$  is a solution of the flow. It is also easy to see, integrating (1.3) on  $M$ , that, for all  $t \in [0, T_1]$ , we have

$$\int_M e^{u(t)} dV = \int_M e^{u_0} dV. \quad (2.1)$$

We also notice that the functional  $E_f(u(t))$  is decreasing with respect to  $t$ . Differentiating  $E_f(u(t))$  with respect to  $t$  and integrating by parts, one finds, for all  $t \in [0, T_1]$ ,

$$\frac{\partial}{\partial t} E_f(u(t)) = - \int_M \left( \frac{\partial u(t)}{\partial t} \right)^2 e^{u(t)} dV \leq 0. \quad (2.2)$$

Therefore, if  $0 \leq t_0 \leq t_1 \leq T_1$ , we have

$$E_f(u(t_1)) \leq E_f(u(t_0)). \quad (2.3)$$

To prove Theorem 1.1, we set

$$T = \sup \left\{ \overline{T} > 0 : \exists u \in C_{loc}^{2+\alpha, 1+\frac{\alpha}{2}}(M \times [0, \overline{T}]) \text{ solution of (1.3)} \right\},$$

and suppose that  $T < +\infty$ . From the definition of  $T$ , we have that  $u \in C_{loc}^{2+\alpha, 1+\frac{\alpha}{2}}(M \times [0, T))$ . We will show that there exists a constant  $C_T > 0$  depending on  $T, M, f, \rho$  and  $\|u_0\|_{C^{2+\alpha}(M)}$  such that

$$\|u\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(M \times [0, T))} \leq C_T. \quad (2.4)$$

This estimate allows us to extend  $u$  beyond  $T$ , contradicting the definition of  $T$ . In the following,  $C$  will denote constants depending on  $M, f, \rho$  and  $\|u_0\|_{C^{2+\alpha}(M)}$  and  $C_T$  the ones depending on  $M, f, \rho, \|u_0\|_{C^{2+\alpha}(M)}$  and  $T$ .

**Proposition 2.1.** *There exists a constant  $C_T$  such that, for all  $t \in [0, T)$ , we have*

$$\|u(t)\|_{H^1(M)} \leq C_T. \quad (2.5)$$

*Proof.* First, we claim that, for all  $t \in [0, T)$ , we have

$$\int_M u^2(t) dV \leq C_1 \int_M |\nabla u(t)|^2 dV + C_2, \quad (2.6)$$

where  $C_1, C_2$  are two constants depending on  $T, f, \rho, \|u_0\|_{C^{2+\alpha}(M)}, M$  and  $A$  (where  $A$  is the set defined in the following). Fix  $t \in [0, T)$  and set

$$M_\varepsilon = \left\{ x \in M : e^{u(x,t)} < \varepsilon \right\},$$

where  $\varepsilon > 0$  is a real number which will be determined later. We set  $\int_M e^{u_0} dV = a$ . Using Hölder's inequality and (2.1), one has

$$\begin{aligned} a = \int_M e^{u(t)} dV &= \int_{M_\varepsilon} e^{u(t)} dV + \int_{M \setminus M_\varepsilon} e^{u(t)} dV \\ &\leq \varepsilon |M_\varepsilon| + |M \setminus M_\varepsilon|^{\frac{1}{2}} \left( \int_M e^{2u(t)} dV \right)^{\frac{1}{2}}. \end{aligned} \quad (2.7)$$

Now, differentiating  $\int_M e^{2u(t)} dV$  with respect to  $t$ , we get

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \left( \int_M e^{2u(t)} dV \right) &= \int_M \Delta u(t) e^{u(t)} dV - \frac{\rho}{|M|} \int_M e^{u(t)} dV + \frac{\rho \int_M f e^{2u(t)} dV}{\int_M f e^{u(t)} dV} \\ &\leq - \int_M |\nabla u(t)|^2 e^{u(t)} dV + C + \frac{\rho \max_{x \in M} f(x)}{a \min_{x \in M} f(x)} \int_M e^{2u(t)} dV \\ &\leq C \int_M e^{2u(t)} dV + C, \quad \forall t \in [0, T). \end{aligned} \quad (2.8)$$

This yields to

$$\int_M e^{2u(t)} dV \leq C_T, \quad \forall t \in [0, T). \quad (2.9)$$

From (2.7), (2.9) and taking  $\varepsilon = \frac{a}{2|M|}$ , we deduce that

$$|M \setminus M_\varepsilon| \geq \left( \frac{a}{2C_T} \right)^2 > 0. \quad (2.10)$$

Using Poincaré's inequality, we have

$$\int_M u^2(t) dV \leq \frac{1}{\lambda_1} \int_M |\nabla u(t)|^2 dV + \frac{1}{|M|} \left( \int_M u(t) dV \right)^2, \quad (2.11)$$

where  $\lambda_1$  is the first eigenvalue of the laplacian. We set  $A = M \setminus M_\varepsilon$ . From (2.10) and since, for all  $x \in M_1$  and  $0 \leq t < T$ ,  $u(x, t) \geq \ln \varepsilon = \ln \left( \frac{a}{2|M|} \right)$ , we deduce that there exists a constant  $C_T$  such that

$$\int_A u(t) dV \geq C_T. \quad (2.12)$$

On the other hand, using (2.1), we have

$$\int_A u(t) dV \leq \int_A e^{u(t)} dV \leq a.$$

We deduce from the previous inequality and (2.12) that there exists a constant  $C_T$  such that

$$\left| \int_A u(t) dV \right| \leq C_T. \quad (2.13)$$

Now, using (2.13) and Young's inequality, we have

$$\begin{aligned} & \frac{1}{|M|} \left( \int_M u(t) dV \right)^2 \\ &= \frac{1}{|M|} \left( \int_A u(t) dV + \int_{M \setminus A} u(t) dV \right)^2 \\ &\leq \frac{1}{|M|} \left( C_T + 2C_T \left| \int_{M \setminus A} u(t) dV \right| + \left( \int_{M \setminus A} u(t) dV \right)^2 \right) \\ &\leq C_T + \frac{2C_T \varepsilon_1 + 1}{|M|} \left( \int_{M \setminus A} u(t) dV \right)^2, \end{aligned} \quad (2.14)$$

where  $\varepsilon_1$  is a strictly positive constant which will be determined later. Using Cauchy-Schwarz's inequality, we obtain

$$\left( \int_{M \setminus A} u(t) dV \right)^2 \leq |M \setminus A| \int_{M \setminus A} u^2(t) dV. \quad (2.15)$$

Combining (2.11), (2.14) and (2.15), we find

$$\begin{aligned} \int_M u^2(t) dV &\leq \frac{1}{\lambda_1} \int_M |\nabla u(t)|^2 dV + C_T \\ &+ \left(1 - \frac{|A|}{|M|} + 2C_T \varepsilon_1 \frac{|M \setminus A|}{|M|}\right) \int_{M \setminus A} u^2(t) dV. \end{aligned}$$

Choosing  $\varepsilon_1$  such that  $\alpha = \left(1 - \frac{|A|}{|M|} + 2C_T \varepsilon_1 \frac{|M \setminus A|}{|M|}\right) < 1$ , we have

$$(1 - \alpha) \int_M u^2(t) dV \leq \frac{1}{\lambda_1} \int_M |\nabla u(t)|^2 dV + C_T.$$

This shows that inequality (2.6) holds. From (2.1) and (2.3), we have

$$C_0 := E_f(u_0) \geq E_f(u(t)) \geq \frac{1}{2} \int_M |\nabla u(t)|^2 dV + \frac{\rho}{|M|} \int_M u(t) dV - C. \quad (2.16)$$

Using Young's inequality, we obtain

$$\frac{1}{2} \int_M |\nabla u(t)|^2 dV \leq C_0 + C + \frac{\rho}{\varepsilon} + \varepsilon \int_M u^2(t) dV,$$

where  $\varepsilon$  is a positive constant which will be determined later. By (2.6), we have

$$\frac{1}{2} \int_M |\nabla u(t)|^2 dV \leq C' + \varepsilon C_1 \int_M |\nabla u(t)|^2 dV,$$

where  $C' = C_0 + C + \frac{\rho}{\varepsilon} + C_2$  and  $C_1, C_2$  are the constants of (2.6). We choose  $\varepsilon$  such that  $\frac{1}{2} - \varepsilon C_1 > 0$ . Consequently, for all  $t \in [0, T]$ , we derive that

$$\int_M |\nabla u(t)|^2 dV \leq C_T. \quad (2.17)$$

Using once more (2.6) and (2.17), we have  $\int_M u^2(t) dV \leq C_T$ . Finally, we conclude that there exists a constant  $C_T > 0$  such that

$$\|u(t)\|_{H^1(M)} \leq C_T, \quad \forall t \in [0, T].$$

□

**Proposition 2.2.** *For all  $\rho \in \mathbb{R}$ , there exists a constant  $C_T$  such that, for all  $t \in [0, T]$ , we have*

$$\|u(t)\|_{H^2(M)} \leq C_T.$$

*Proof.* In view of Proposition 2.1, we just need to bound  $\int_M (\Delta u(t))^2 dV$ , for all  $t \in [0, T]$ . We begin by setting

$$v(t) = \frac{\partial u(t)}{\partial t} e^{\frac{u(t)}{2}},$$

then equation (1.3) becomes

$$v(t)e^{\frac{u(t)}{2}} = \Delta u(t) - \frac{\rho}{|M|} + \frac{\rho f e^{u(t)}}{\int_M f e^{u(t)} dV}.$$

Differentiating  $\int_M (\Delta u(t))^2 dV$  with respect to  $t$  and integrating by parts on  $M$ , we have

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial t} \int_M (\Delta u(t))^2 dV \\ &= \int_M \left( v(t)e^{\frac{u(t)}{2}} + \frac{\rho}{|M|} - \frac{\rho f e^{u(t)}}{\int_M f e^{u(t)} dV} \right) \Delta \left( v(t)e^{-\frac{u(t)}{2}} \right) dV \\ &= - \int_M |\nabla v(t)|^2 dV + \frac{1}{4} \int_M v^2(t) |\nabla u(t)|^2 dV \\ &+ \frac{\rho}{\int_M f e^{u(t)} dV} \left( \int_M \nabla f \nabla v(t) e^{\frac{u(t)}{2}} dV - \frac{1}{2} \int_M \nabla f v(t) \nabla u(t) e^{\frac{u(t)}{2}} dV \right. \\ &\left. + \int_M f \nabla u(t) \nabla v(t) e^{\frac{u(t)}{2}} dV - \frac{1}{2} \int_M f |\nabla u(t)|^2 v(t) e^{\frac{u(t)}{2}} dV \right). \end{aligned}$$

Since  $f \in C^\infty(M)$  and is strictly positive (in particular we have  $\int_M f e^{u(t)} dV \geq C \min_{x \in M} f(x)$ ), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial t} \int_M (\Delta u(t))^2 dV \\ &\leq - \int_M |\nabla v(t)|^2 dV + C \int_M v^2(t) |\nabla u(t)|^2 dV \\ &+ C \left( \int_M e^{\frac{u(t)}{2}} \left( |\nabla v(t)| + |v(t)| |\nabla u(t)| + |\nabla u(t)| |\nabla v(t)| + |\nabla u(t)|^2 |v(t)| \right) dV \right). \end{aligned} \tag{2.18}$$

Let's estimate the positive terms on the right side of (2.18). From Hölder's inequality, we have, recalling that  $\int_M e^{u(t)} dV = \int_M e^{u_0} dV$ , for all  $t \in [0, T)$ ,

$$\int_M |\nabla v(t)| e^{\frac{u(t)}{2}} dV \leq C \|v(t)\|_{H^1(M)}^{\frac{1}{2}}. \tag{2.19}$$

Using Gagliardo-Nirenberg's inequality (see for example [3])

$$\|h\|_{L^4(M)}^2 \leq C \|h\|_{L^2(M)} \|h\|_{H^1(M)}, \quad \forall h \in H^1(M),$$

and Cauchy-Schwarz's inequality, we have

$$\begin{aligned} \int_M v^2(t) |\nabla u(t)|^2 dV &\leq \left( \int_M v^4(t) dV \right)^{\frac{1}{2}} \left( \int_M |\nabla u(t)|^4 dV \right)^{\frac{1}{2}} \\ &= \|v(t)\|_{L^4(M)}^2 \|\nabla u(t)\|_{L^4(M)}^2 \\ &\leq C \|v(t)\|_{L^2(M)} \|v(t)\|_{H^1(M)} \|\nabla u(t)\|_{L^2(M)} \|\nabla u(t)\|_{H^1(M)}. \end{aligned}$$

Since, by Proposition 2.1,  $\|u(t)\|_{H^1(M)} \leq C_T$  for all  $t \in [0, T)$ , we get

$$\int_M v^2(t) |\nabla u(t)|^2 dV \leq C_T \|v(t)\|_{L^2(M)} \|v(t)\|_{H^1(M)} \|u(t)\|_{H^2(M)} \quad (2.20)$$

Using Proposition 2.1 and Moser-Trudinger inequality (3.2), we deduce that there exists a constant  $C_T$  such that, for all  $t \in [0, T)$ , and  $p \in \mathbb{R}$ ,

$$\int_M e^{pv(t)} dV \leq C_T. \quad (2.21)$$

In the same way as to prove (2.20) and using (2.21), we obtain

$$\begin{aligned} & \int_M |\nabla u(t)| |v(t)| e^{\frac{u(t)}{2}} dV \\ & \leq \left( \int_M |\nabla u(t)|^2 dV \right)^{\frac{1}{2}} \left( \int_M v^4(t) dV \right)^{\frac{1}{4}} \left( \int_M e^{2u(t)} dV \right)^{\frac{1}{4}} \\ & \leq C_T \|v(t)\|_{L^4(M)} \\ & \leq C_T \|v(t)\|_{L^2(M)}^{\frac{1}{2}} \|v(t)\|_{H^1(M)}^{\frac{1}{2}}, \end{aligned} \quad (2.22)$$

$$\begin{aligned} & \int_M |\nabla u(t)| |\nabla v(t)| e^{\frac{u(t)}{2}} dV \\ & \leq \left( \int_M |\nabla v(t)|^2 dV \right)^{\frac{1}{2}} \left( \int_M |\nabla u(t)|^4 dV \right)^{\frac{1}{4}} \left( \int_M e^{2u(t)} dV \right)^{\frac{1}{4}} \\ & \leq C_T \|v(t)\|_{H^1(M)} \|u(t)\|_{H^2(M)}^{\frac{1}{2}}, \end{aligned} \quad (2.23)$$

and

$$\begin{aligned} & \int_M |\nabla u(t)|^2 |v(t)| e^{\frac{u(t)}{2}} dV \\ & \leq C_T \|v(t)\|_{L^4(M)} \|\nabla u(t)\|_{L^4(M)}^2 \\ & \leq C_T \|u(t)\|_{H^2(M)} \|v(t)\|_{L^2(M)}^{\frac{1}{2}} \|v(t)\|_{H^1(M)}^{\frac{1}{2}}. \end{aligned} \quad (2.24)$$

Finally, inserting estimates (2.19), (2.20), (2.22), (2.23), (2.24) into (2.18), it follows that

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \int_M (\Delta u(t))^2 dV & \leq - \int_M |\nabla v(t)|^2 dV + C_T \|v(t)\|_{L^2(M)} \|v(t)\|_{H^1(M)} \|u(t)\|_{H^2(M)} \\ & \quad + C_T \left( \|v(t)\|_{H^1(M)}^{\frac{1}{2}} + \|v(t)\|_{H^1(M)}^{\frac{1}{2}} \|v(t)\|_{L^2(M)}^{\frac{1}{2}} \right) \\ & \quad + C_T \|v(t)\|_{H^1(M)} \|u(t)\|_{H^2(M)}^{\frac{1}{2}} \\ & \quad + C_T \|u(t)\|_{H^2(M)} \|v(t)\|_{L^2(M)}^{\frac{1}{2}} \|v(t)\|_{H^1(M)}^{\frac{1}{2}}. \end{aligned}$$

Using Young's inequality on each positive terms, we obtain

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \int_M (\Delta u(t))^2 dV + 1 \right) \\ & \leq C_T \left( \int_M (\Delta u(t))^2 dV + 1 \right) \left( \|v(t)\|_{L^2(M)} + 1 \right). \end{aligned} \quad (2.25)$$



On the other hand, for all  $t \in [0, T]$ , one has, since  $\|u(t)\|_{H^1(M)} \leq C_T$ ,

$$\begin{aligned}
\int_0^t \|v(s)\|_{L^2(M)}^2 ds &= \int_0^t \int_M v^2(s) dV ds \\
&= \int_0^t \int_M \left( \frac{\partial u(s)}{\partial t} \right)^2 e^{u(s)} dV ds \\
&= - \int_0^t \frac{\partial}{\partial t} E_f(u(s)) ds = E_f(u_0) - E_f(u(t)) \\
&\leq C_T.
\end{aligned} \tag{2.26}$$

Thus, integrating (2.25) with respect to  $t$  and using (2.26), it follows that

$$\int_M (\Delta u(t))^2 dV \leq C_T, \quad \forall t \in [0, T].$$

Therefore we conclude that

$$\|u(t)\|_{H^2(M)} \leq C_T, \quad \forall t \in [0, T].$$

□

*Proof of Theorem 1.1.* We recall that to prove Theorem 1.1, it is sufficient to prove (2.4), i.e. there exists a constant  $C_T > 0$  such that

$$\|u\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(M \times [0, T])} \leq C_T.$$

First, we claim that for all  $\alpha \in (0, 1)$ , there exists a constant  $C_T$  such that

$$|u(x_1, t_1) - u(x_2, t_2)| \leq C_T(|t_1 - t_2|^{\frac{\alpha}{2}} + |x_1 - x_2|^\alpha), \tag{2.27}$$

for all  $x_1, x_2 \in M$  and all  $t_1, t_2 \in [0, T]$ , where  $|x_1 - x_2|$  stands for the geodesic distance from  $x_1$  to  $x_2$  with respect to the metric  $g$ . From Proposition 2.2 and Sobolev's embedding Theorem, we get, for  $\alpha \in (0, 1)$ ,  $t \in [0, T]$  that there exists a constant  $C_T$  such that  $\|u(t)\|_{C^\alpha(M)} \leq C_T$ , i.e. for all  $x, y \in M$ ,

$$|u(x, t) - u(y, t)| \leq C_T |x - y|^\alpha. \tag{2.28}$$

If  $t_2 - t_1 \geq 1$ , using (2.28), it is easy to see that (2.27) holds. Therefore, from now on, we assume that  $0 < t_2 - t_1 < 1$ . On the other hand, since  $u(t)$  is a solution of (1.3) and  $\|u(t)\|_{C^\alpha(M)} \leq C_T$ , one has,  $\forall t \in [0, T]$ ,

$$\left| \frac{\partial u(t)}{\partial t} \right|^2 \leq C_T |\Delta u(t)|^2 + C_T.$$

Integrating the previous estimate on  $M$ , we obtain, for all  $t \in [0, T]$ ,

$$\int_M \left| \frac{\partial u(t)}{\partial t} \right|^2 dV \leq C_T \|u(t)\|_{H^2(M)}^2 + C_T \leq C_T. \tag{2.29}$$

Now, we write

$$\begin{aligned}
|u(x, t_1) - u(x, t_2)| &= \frac{1}{|B_{\sqrt{t_2-t_1}}(x)|} \int_{B_{\sqrt{t_2-t_1}}(x)} |u(x, t_1) - u(x, t_2)| dV(y) \\
&\leq \frac{C}{t_2 - t_1} \int_{B_{\sqrt{t_2-t_1}}(x)} |u(x, t_1) - u(y, t_1)| dV(y) \\
&+ \frac{C}{t_2 - t_1} \int_{B_{\sqrt{t_2-t_1}}(x)} |u(y, t_1) - u(y, t_2)| dV(y) \\
&+ \frac{C}{t_2 - t_1} \int_{B_{\sqrt{t_2-t_1}}(x)} |u(y, t_2) - u(x, t_2)| dV(y), \tag{2.30}
\end{aligned}$$

where  $B_{\sqrt{t_2-t_1}}(x)$  stands for the geodesic ball of radius  $\sqrt{t_2 - t_1}$  centered in  $x$ . Let's consider the first term on the right of (2.30). Using (2.28), we obtain

$$\begin{aligned}
&\frac{C}{t_2 - t_1} \int_{B_{\sqrt{t_2-t_1}}(x)} |u(x, t_1) - u(y, t_1)| dV(y) \\
&\leq \frac{C_T}{(t_2 - t_1)} \int_{B_{\sqrt{t_2-t_1}}(x)} |x - y|^\alpha dV(y) \\
&\leq C_T (t_2 - t_1)^{\frac{\alpha}{2}}. \tag{2.31}
\end{aligned}$$

In the same way, we have

$$\frac{C}{t_2 - t_1} \int_{B_{\sqrt{t_2-t_1}}(x)} |u(x, t_2) - u(y, t_2)| dV(y) \leq C_T (t_2 - t_1)^{\frac{\alpha}{2}}. \tag{2.32}$$

We have, using Hölder's inequality and (2.29),

$$\begin{aligned}
&\frac{C}{t_2 - t_1} \int_{B_{\sqrt{t_2-t_1}}(x)} |u(y, t_1) - u(y, t_2)| dV(y) \\
&\leq C \sup_{t_1 \leq \tau \leq t_2} \int_{B_{\sqrt{t_2-t_1}}(x)} \left| \frac{\partial u}{\partial s} \right| (y, \tau) dV(y) \\
&\leq C \sqrt{t_2 - t_1} \sup_{t_1 \leq \tau \leq t_2} \left( \int_{B_{\sqrt{t_2-t_1}}(x)} \left| \frac{\partial u}{\partial s} \right|^2 (y, \tau) dV(y) \right)^{\frac{1}{2}} \\
&\leq C_T \sqrt{t_2 - t_1}. \tag{2.33}
\end{aligned}$$

Putting (2.31), (2.32), (2.33) in (2.30) and noticing that for all  $0 < t_2 - t_1 < 1$ , we have  $\sqrt{t_2 - t_1} \leq (t_2 - t_1)^{\frac{\alpha}{2}}$ , we find

$$|u(x, t_1) - u(x, t_2)| \leq C_T (t_2 - t_1)^{\frac{\alpha}{2}}. \tag{2.34}$$

The inequalities (2.28) and (2.34) imply that

$$|u(x_1, t_1) - u(x_2, t_2)| \leq C_T (|t_1 - t_2|^{\frac{\alpha}{2}} + |x_1 - x_2|^\alpha),$$

for all  $x_1, x_2 \in M$  and all  $t_1, t_2 \in [0, T)$ ,  $0 < t_2 - t_1 < 1$ . This establishes (2.27). In view of (2.27), we may apply the standard regularity theory for

parabolic equations (see for example [11]) to derive that there exists a constant  $C_T$  depending on  $T$  such that

$$\|u\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(M \times [0, T])} \leq C_T, \quad \alpha \in (0, 1).$$

This completes the proof of Theorem 1.1.

### 3 Improved Moser-Trudinger's inequality.

We begin by recalling the Moser-Trudinger's inequality (see [15], [21]): there exists a constant  $C$  depending on  $(M, g)$  such that, for all  $u \in H^1(M)$ ,

$$\int_M e^{\frac{4\pi(u-\bar{u})^2}{|\nabla u|^2 dV}} dV \leq C, \quad (3.1)$$

where  $\bar{u}$  stands for the mean value of  $u$  on  $M$  i.e.  $\bar{u} = \frac{\int_M u dV}{|M|}$ . As a consequence of (3.1), we obtain the following inequality : there exists a constant  $C$  depending on  $(M, g)$  such that, for all  $u \in H^1(M)$ ,

$$\log \left( \int_M e^{u-\bar{u}} dV \right) \leq \frac{1}{16\pi} \int_M |\nabla u|^2 dV + C. \quad (3.2)$$

The next lemma shows that the Moser-Trudinger's inequality (3.2) can be improved for functions which "concentrate" in some points.

**Lemma 3.1** ([7]). *Let  $\delta_0, \gamma_0$  be some positive real numbers,  $l$  an integer,  $\Omega_1, \dots, \Omega_l$  subsets of  $M$  such that  $\text{dist}(\Omega_i, \Omega_j) \geq \delta_0$  for  $i \neq j$ . Then, for all  $\tilde{\varepsilon} > 0$ , there exists a constant  $C$  depending on  $l, \tilde{\varepsilon}, \delta_0$  and  $\gamma_0$  such that*

$$\log \left( \int_M e^{u-\bar{u}} dV \right) \leq \left( \frac{1}{16l\pi} + \tilde{\varepsilon} \right) \int_M |\nabla u|^2 dV + C,$$

for  $u \in H^1(M)$  such that, for all  $i \in \{1, \dots, l\}$ ,

$$\int_{\Omega_i} e^u dV \geq \gamma_0 \int_M e^u dV.$$

We define  $H_G^1(M)$  the space of functions  $u \in H^1(M)$  such that  $u$  is  $G$ -invariant. By considering functions  $u \in H_G^1(M)$  and assuming hypothesis on the cardinal of orbits of points of  $M$  under the action of  $G$ , we find that there exist subsets  $\{\Omega_i\}_{1 \leq i \leq l}$  of  $M$  such as described previously. More precisely, we have

**Proposition 3.1.** *Let  $k \in \mathbb{N}$ ,  $k \geq 2$ . Suppose that  $\min_{x \in M} |O_G(x)| \geq k$ . Then, for all  $\varepsilon > 0$ , there exists a constant  $C$  positive depending on  $M$  and  $\varepsilon$  such that, for all  $u \in H_G^1(M)$ ,*

$$\log \left( \int_M e^{u-\bar{u}} dV \right) \leq \left( \frac{1}{16k\pi} + \varepsilon \right) \int_M |\nabla u|^2 dV + C. \quad (3.3)$$

*Proof.* The proof will be divided into two steps.

**Step 1.** Let  $u \in H_G^1(M)$  then there exists  $x_1 \in M$  (depending on  $u$ ) such that,  $\forall 0 < r < i(M)$ ,

$$\int_{B_r(x_1)} e^u dV \geq Cr^2 \int_M e^u dV,$$

where  $i(M)$  stands for the injectivity radius of  $(M, g)$ ,  $B_r(x_i)$  stands for the geodesic ball of radius  $r$  centered in  $x_i$  and  $C$  is a positive constant depending on  $(M, g)$ .

*Proof of Step 1.* Let  $0 < r < i(M)$ . Since  $M$  is compact, there exists a finite number of points  $x_1, \dots, x_m \in M$  such that

$$M = \bigcup_{i=1}^m B_r(x_i) ; B_{\frac{r}{2}}(x_i) \cap B_{\frac{r}{2}}(x_j) = \emptyset, \forall i \neq j.$$

We can assume, up to relabelling the  $x_i$ 's,  $i \in \{1, \dots, m\}$ , that

$$\int_{B_r(x_1)} e^u dV = \max_{i \in \{1, \dots, m\}} \int_{B_r(x_i)} e^u dV.$$

Therefore, we have

$$\int_M e^u dV \leq \sum_{i=1}^m \int_{B_r(x_i)} e^u dV \leq m \int_{B_r(x_1)} e^u dV. \quad (3.4)$$

On the other hand, there exists  $C$ , a positive constant depending only on  $(M, g)$  such that, for all  $1 \leq i \leq m$ ,

$$|B_{\frac{r}{2}}(x_i)| \geq Cr^2,$$

thus

$$|M| \geq \sum_{i=1}^m |B_{\frac{r}{2}}(x_i)| \geq mCr^2. \quad (3.5)$$

Combining (3.4) and (3.5), we obtain

$$\int_{B_r(x_1)} e^u dV \geq \frac{Cr^2}{|M|} \int_M e^u dV.$$

**Step 2.** Let  $G$  be an isometry group such that, for all  $x \in M$ ,  $|O_G(x)| \geq k$ . There exists  $\delta$  depending on  $M, G$  and  $k$  such that, for all  $x \in M$ , there exist  $k$  points  $x = x_1, \dots, x_k \in O_G(x)$  such that

$$|x_i - x_j| \geq \delta \text{ si } i \neq j,$$

where  $|x_i - x_j|$  stands for the geodesic distance between  $x_i$  and  $x_j$  with respect to the metric  $g$ .

*Proof of Step 2.* We proceed by induction on  $k \geq 2$ . Suppose that  $k = 2$ . By contradiction, we assume that  $\forall n \in \mathbb{N}^*, \exists x_n \in M$  such that

$$|x_n - \sigma(x_n)| < \frac{1}{n}, \forall \sigma \in G.$$

Since  $M$  is compact, there exists  $x \in M$  such that  $|x_n - x| \xrightarrow{n \rightarrow +\infty} 0$ . Therefore, we find  $|x - \sigma(x)| = 0$  for all  $\sigma \in G$ . This implies that  $|O_G(x)| = 1$ . This provides us a contradiction. Now suppose that the induction hypothesis holds for groups  $G$  such that  $|O_G(x)| \geq k$  for all  $x \in M$ , i.e.

$$\begin{aligned} \exists \delta_k > 0 \text{ s.t. } \forall x \in M, \exists k \text{ points } x = x_1, \dots, x_k \in O_G(x) \\ \text{such that } |x_i - x_j| \geq \delta_k, \quad \forall i \neq j \in \{1, \dots, k\}. \end{aligned} \quad (3.6)$$

Let's consider  $G$  a group such that  $|O_G(x)| \geq k + 1$  for all  $x \in M$ . Proceeding by contradiction, from (3.6), we see that  $\forall n \in \mathbb{N}^*, \exists x_n^1 \in M$  and  $x_n^2, \dots, x_n^k \in O_G(x_n^1)$  such that

$$|x_n^i - x_n^j| \geq \delta_k, \quad \forall i \neq j \in \{1, \dots, k\},$$

and,  $\forall \sigma \in G$ ,

$$\inf_{j \in \{1, \dots, k\}} |\sigma(x_n^1) - x_n^j| < \frac{1}{n}. \quad (3.7)$$

Since  $M$  is compact, we can assume that there exist  $x^1, \dots, x^k \in M$  such that

$$|x_n^j - x^j| \xrightarrow{n \rightarrow +\infty} 0, \quad \forall j \in \{1, \dots, k\}.$$

Letting  $n$  tend to  $+\infty$  in (3.7), we deduce that,  $\forall \sigma \in G$ ,

$$\inf_{j \in \{1, \dots, k\}} |\sigma(x^1) - x^j| = 0.$$

This implies that

$$O_G(x^1) \subseteq \{x^1, \dots, x^k\}.$$

Therefore we get a contradiction with the fact that  $|O_G(x)| \geq k + 1$  for all  $x \in M$ .

*Proof of Lemma 3.1.* Let  $u \in H_G^1(M)$ . From Step 1, there exists  $x_1 \in M$  such that, for all  $0 < r < i(M)$ ,

$$\int_{B_r(x_1)} e^u dV \geq Cr^2 \int_M e^u dV.$$

By Step 2, there exist a constant  $\delta > 0$  depending on  $M, k, G$  and points  $x_2, \dots, x_k \in O_G(x_1)$  such that  $|x_i - x_j| \geq \delta$ , for all  $i \neq j \in \{1, \dots, k\}$ . We can suppose that  $\delta < i(M)$ . Setting  $\Omega_i = B_{\frac{\delta}{4}}(x_i)$ , we have

$$\text{dist}(\Omega_i, \Omega_j) \geq \frac{\delta}{2}. \quad (3.8)$$

Since  $x_j \in O_G(x_1)$  for all  $j \in \{2, \dots, k\}$ , there exists  $\sigma_j \in G$  such that  $x_j = \sigma_j(x_1)$ . Using the  $G$ -invariance of  $u$ , we have, for all  $j \in \{2, \dots, k\}$ ,

$$\int_{\Omega_j} e^u dV = \int_{\Omega_1} e^u dV \geq C\delta^2 \int_M e^u dV. \quad (3.9)$$

From (3.8) and (3.9), the hypothesis of Lemma 3.1 are satisfied. The proof follows immediately.  $\square$

## 4 Proof of Theorem 1.2.

This section is devoted to the convergence of the flow when its initial data  $u_0 \in C^{2+\alpha}(M)$ ,  $\alpha \in (0, 1)$ , and the function  $f \in C^\infty(M)$  are invariant under the action of an isometry group  $G$  acting on  $(M, g)$ . Let  $u : M \times [0, +\infty) \rightarrow \mathbb{R}$  be the unique global solution of (1.3). We begin by noticing, since  $u_0 \in C_G^{2+\alpha}(M)$ ,  $\alpha \in (0, 1)$ , and  $f \in C_G^\infty$ , that  $u(t)$  is  $G$ -invariant for all  $t \geq 0$ . First, we prove that  $u(t)$ ,  $t \geq 0$ , is uniformly (in time) bounded in  $H^1(M)$ . In the following,  $k$  will stand for  $\min_{x \in M} |O_G(x)|$  and  $C$  will denote constants depending on  $M$ ,  $f$ ,  $\rho$  and  $\|u_0\|_{C^{2+\alpha}(M)}$ .

**Proposition 4.1.** *Let  $\rho < 8k\pi$ . Then, there exists a constant  $C$  such that*

$$\|u(t)\|_{H^1(M)} \leq C, \quad \forall t \geq 0. \quad (4.1)$$

*Proof.* From (2.3), we have

$$E_f(u(t)) \leq E_f(u_0) := C_0, \quad \forall t \geq 0. \quad (4.2)$$

To prove (4.1), we will consider two cases  $\rho < 0$  and  $0 \leq \rho < 8k\pi$ . In a first time, let's consider the case  $\rho < 0$ . Since  $f \in C^\infty(M)$  and is a strictly positive function, we have

$$\begin{aligned} E_f(u(t)) &= \frac{1}{2} \int_M |\nabla u(t)|^2 dV + \frac{\rho}{|M|} \int_M u(t) dV - \rho \log \left( \int_M f e^{u(t)} dV \right) \\ &\geq \frac{1}{2} \int_M |\nabla u(t)|^2 dV - \rho \log \left( \int_M e^{u(t) - \bar{u}(t)} dV \right) - C. \end{aligned}$$

Using Jensen's inequality, it follows that

$$E_f(u(t)) \geq \frac{1}{2} \int_M |\nabla u(t)|^2 dV - C.$$

From (4.2), we find

$$\int_M |\nabla u(t)|^2 dV \leq C, \quad \forall t \geq 0. \quad (4.3)$$

Now let's consider the second case  $0 \leq \rho < 8k\pi$ . Since, as we already notice,  $u(t)$  is  $G$ -invariant, for all  $t \geq 0$ , we can use the improved Moser-Trudinger's inequality (3.3) of Lemma 3.1. This gives us

$$\begin{aligned} E_f(u(t)) &\geq \frac{1}{2} \int_M |\nabla u(t)|^2 dV - \rho \log \left( \int_M e^{u(t) - \bar{u}(t)} dV \right) - C \\ &\geq \left( \frac{1}{2} - \frac{\rho}{16k\pi} - \varepsilon \right) \int_M |\nabla u(t)|^2 dV - C. \end{aligned} \quad (4.4)$$

Since  $0 \leq \rho < 8k\pi$ , we obtain, taking  $\varepsilon = \frac{8k\pi - \rho}{32k\pi}$ , using (4.2) and (4.4),

$$\int_M |\nabla u(t)|^2 dV \leq C, \quad \forall t \geq 0. \quad (4.5)$$

From (4.3) and (4.5), we deduce that, for all  $\rho < 8k\pi$ ,

$$\int_M |\nabla u(t)|^2 dV \leq C, \quad \forall t \geq 0. \quad (4.6)$$

Now, using Poincaré's inequality, we get

$$\|u(t) - \bar{u}(t)\|_{H^1(M)} \leq C. \quad (4.7)$$

From the improved Moser-Trudinger's inequality (3.3) and (4.6), we have

$$\int_M e^{u(t) - \bar{u}(t)} dV \leq C.$$

Since,  $\int_M e^{u(t)} dV = \int_M e^{u_0} dV$ , for all  $t \geq 0$ , we find

$$\bar{u}(t) \geq -C, \quad \forall t \geq 0. \quad (4.8)$$

On the other hand, using Jensen's inequality, we have

$$\bar{u}(t) \leq -C, \quad \forall t \geq 0. \quad (4.9)$$

Finally, from (4.7), (4.8) and (4.9), we conclude that

$$\|u(t)\|_{H^1(M)} \leq C, \quad \forall t \geq 0. \quad (4.10)$$

From (3.3) and (4.10), we deduce that there exists a constant  $C$  (not depending on time) such that, for all  $t \geq 0$  and  $p \in \mathbb{R}$ ,

$$\int_M e^{pu(t)} dV \leq C. \quad (4.11)$$

We also notice that integrating  $\frac{\partial}{\partial t} E_f(u(t)) = - \int_M \left| \frac{\partial}{\partial t} u(t) \right|^2 e^{u(t)} dV$  with respect to  $t$  and since  $\|u(t)\|_{H^1(M)} \leq C$ , we find, for all  $T_1 \geq 0$ ,

$$\int_0^{T_1} \int_M \left| \frac{\partial}{\partial t} u(t) \right|^2 e^{u(t)} dV dt = E_f(u_0) - E_f(u(T_1)) \leq C. \quad (4.12)$$

□

*Proof of Theorem 1.2.* We follow closely Brendle [2] arguments. We set  $U(t) = \frac{\partial}{\partial t} u(t)$  and  $y(t) = \int_M U^2(t) e^{u(t)} dV$ . We claim that

$$y(t) \xrightarrow{t \rightarrow +\infty} 0.$$

Let  $\varepsilon$  be some real positive number. In view of (4.12), there exists  $t_0 > 0$  such that  $y(t_0) \leq \varepsilon$ . We want to show that

$$y(t) \leq 3\varepsilon, \quad \forall t \geq t_0. \quad (4.13)$$

Suppose, by contradiction, that (4.13) doesn't hold. We set

$$t_1 = \inf \{ t \geq t_0 : y(t) \geq 3\varepsilon \} < +\infty.$$

By definition, we have

$$y(t) \leq 3\varepsilon, \quad \forall t_0 \leq t \leq t_1.$$

Since  $\frac{\partial u(t)}{\partial t} = e^{-u(t)} \left( \Delta u(t) - \frac{\rho}{|M|} \right) + \frac{\rho f}{\int_M f e^{u(t)} dV}$ , we find, using Young's inequality, for all  $t_0 \leq t \leq t_1$ ,

$$\begin{aligned} & \int_M e^{-u(t)} \left( \Delta u(t) - \frac{\rho}{|M|} \right)^2 dV \\ &= y(t) - 2 \frac{\rho}{\int_M f e^{u(t)} dV} \int_M f \frac{\partial u}{\partial t}(t) e^{u(t)} dV + \int_M \left( \frac{\rho f}{\int_M f e^{u(t)} dV} \right)^2 e^{u(t)} dV \\ &\leq C y(t) + C \int_M e^{u(t)} dV \leq C_1, \end{aligned} \quad (4.14)$$

where  $C_1$  is a constant depending on  $t_1$  and  $C$  is a constant not depending on time. From (4.11) with  $p = 3$ , we have, for all  $t \geq 0$ ,

$$\int_M e^{3u(t)} dV \leq C. \quad (4.15)$$

Using Hölder's inequality, (4.14) and (4.15), we obtain, for all  $t_0 \leq t \leq t_1$ ,

$$\begin{aligned} \int_M \left| \Delta u(t) - \frac{\rho}{|M|} \right|^{\frac{3}{2}} dV &\leq \left( \int_M e^{-u(t)} \left( \Delta u(t) - \frac{\rho}{|M|} \right)^2 dV \right)^{\frac{3}{4}} \left( \int_M e^{3u(t)} dV \right)^{\frac{1}{4}} \\ &\leq C_1, \end{aligned}$$

therefore

$$\int_M |\Delta u(t)|^{\frac{3}{2}} dV \leq C_1, \quad \forall t_0 \leq t \leq t_1. \quad (4.16)$$

We deduce from the Sobolev's embedding Theorem that

$$|u(t)| \leq C_1, \quad \forall t_0 \leq t \leq t_1. \quad (4.17)$$

Derivating (1.3) with respect to  $t$ , we see that  $U(t) = \frac{\partial u(t)}{\partial t}$  satisfies

$$\begin{aligned} \frac{\partial U(t)}{\partial t} &= e^{-u(t)} \Delta U(t) - U(t) e^{-u(t)} \Delta u(t) \\ &\quad + \frac{\rho}{|M|} U(t) e^{-u(t)} - \frac{\rho f}{\left( \int_M f e^{u(t)} dV \right)^2} \int_M U(t) f e^{u(t)} dV. \end{aligned} \quad (4.18)$$

Now, using (4.18), we have

$$\begin{aligned} \frac{\partial y(t)}{\partial t} &= \frac{\partial}{\partial t} \left( \int_M U^2(t) e^{u(t)} dV \right) \\ &= 2 \int_M U(t) e^{u(t)} \left( e^{-u(t)} \Delta U(t) - U(t) e^{-u(t)} \Delta u(t) + \frac{\rho}{|M|} U(t) e^{-u(t)} \right. \\ &\quad \left. - \frac{\rho f}{\left( \int_M f e^{u(t)} dV \right)^2} \int_M U(t) f e^{u(t)} dV \right) dV + \int_M U^3(t) e^{u(t)} dV. \end{aligned}$$



Integrating by parts on  $M$  and since  $\Delta u(t) - \frac{\rho}{|M|} = U(t)e^{u(t)} - \frac{\rho f e^{u(t)}}{\int_M f e^{u(t)} dV}$ , we obtain

$$\begin{aligned}
\frac{\partial y(t)}{\partial t} &= -2 \int_M |\nabla U(t)|^2 dV - 2 \int_M U^2(t) \left( \Delta u(t) - \frac{\rho}{|M|} \right) dV \\
&- \frac{2\rho}{\left( \int_M f e^{u(t)} dV \right)^2} \left( \int_M f U(t) e^{u(t)} dV \right)^2 + \int_M U^3(t) e^{u(t)} dV \\
&= -2 \int_M |\nabla U(t)|^2 dV - \int_M U^3(t) e^{u(t)} dV \\
&+ 2\rho \left( \frac{\int_M f U^2(t) e^{u(t)} dV}{\int_M f e^{u(t)} dV} - \left( \frac{\int_M f U(t) e^{u(t)} dV}{\int_M f e^{u(t)} dV} \right)^2 \right). \quad (4.19)
\end{aligned}$$

Using the Gagliardo-Nirenberg's inequality, we get

$$\|U(t)\|_{L^3_{g_1(t)}(M)} \leq C \|U(t)\|_{L^2_{g_1(t)}(M)}^{\frac{2}{3}} \|U(t)\|_{H^1_{g_1(t)}(M)}^{\frac{1}{3}}, \quad (4.20)$$

where the norm are taken with respect to the conformal metric  $g_1(t) = e^{u(t)}g$ . Let's  $\tilde{\lambda}_1(t)$  be the first eigenvalue of the laplacian with respect to the metric  $g_1(t)$ . By the Rayleigh quotient, we have

$$\tilde{\lambda}_1(t) = \inf_{v \in H^1_{g_1(t)}(M)} \frac{\int_M |\nabla^{g_1(t)} v|_{g_1(t)}^2 dV_{g_1(t)}}{\int_M v^2 dV_{g_1(t)}}.$$

From (4.17) and since  $\int_M |\nabla^{g_1(t)} v|_{g_1(t)}^2 dV_{g_1(t)} = \int_M |\nabla v|^2 dV$ , we deduce that,  $\forall t_0 \leq t \leq t_1$ ,

$$\tilde{\lambda}_1(t) \geq C_1. \quad (4.21)$$

From Poincaré's inequality, (4.21) and since  $\int_M U(t) e^{u(t)} dV = 0$ , we find,  $\forall t_0 \leq t \leq t_1$ ,

$$\int_M U^2(t) e^{u(t)} dV \leq \frac{1}{\tilde{\lambda}_1(t)} \int_M |\nabla U(t)|^2 dV \leq C_1 \int_M |\nabla U(t)|^2 dV,$$

hence, we get

$$\|U(t)\|_{H^1_{g_1(t)}(M)} \leq C_1 \left( \int_M |\nabla U(t)|^2 dV \right)^{\frac{1}{2}}. \quad (4.22)$$

Inserting (4.22) into (4.20), we obtain

$$\int_M e^{u(t)} |U(t)|^3 dV \leq C_1 \left( \int_M U^2(t) e^{u(t)} dV \right) \left( \int_M |\nabla U(t)|^2 dV \right)^{\frac{1}{2}}.$$

Putting the previous estimate into (4.19) and using Young and Hölder's inequal-

ity, we find

$$\begin{aligned}
\frac{\partial y(t)}{\partial t} &\leq -2 \int_M |\nabla U(t)|^2 dV + C_1 \left( \int_M U^2(t) e^{u(t)} dV \right) \left( \int_M |\nabla U(t)|^2 dV \right)^{\frac{1}{2}} \\
&+ C \int_M U^2(t) e^{u(t)} dV + C \left( \int_M |U(t)| e^{u(t)} dV \right)^2 \\
&\leq C_1 \left( \int_M U^2(t) e^{u(t)} dV \right)^2 + C \left( \int_M U^2(t) e^{u(t)} dV \right).
\end{aligned}$$

This implies, since  $y(t_0) \leq \varepsilon$  and  $y(t_1) = 3\varepsilon$ , that

$$2\varepsilon \leq y(t_1) - y(t_0) \leq C_1 \int_{t_0}^{t_1} y(t) dt.$$

Choosing  $t_0$  sufficiently large, i.e. such that

$$C_1 \int_{t_0}^{+\infty} y(t) dt \leq \varepsilon,$$

we obtain a contradiction. Thus, we have established that

$$y(t) = \int_M \left( \frac{\partial u}{\partial t}(t) \right)^2 e^{u(t)} dV \xrightarrow[t \rightarrow +\infty]{} 0. \quad (4.23)$$

Moreover, we get that all estimates we found during the proof, hold for all  $t \geq 0$ . In particular, we have that

$$|u(t)| \leq C, \quad \forall t \geq 0.$$

Thus, from the previous estimate, (4.23) and using Young's inequality, we deduce that

$$\begin{aligned}
\int_M \left( \Delta u(t) - \frac{\rho}{|M|} \right)^2 dV &= \int_M \left( \frac{\partial e^{u(t)}}{\partial t} - \frac{f e^{u(t_n)}}{\int_M f e^{u(t_n)} dV} \right)^2 \\
&\leq C y(t) + C \leq C, \quad \forall t \geq 0.
\end{aligned}$$

This implies that

$$\|u(t)\|_{H^2(M)} \leq C, \quad \forall t \geq 0.$$

Therefore, there exist  $u_\infty \in H^2(M)$  and a sequence  $(t_n)_n$ ,  $t_n \xrightarrow[n \rightarrow +\infty]{} +\infty$  such that

$$u(t_n) \xrightarrow[n \rightarrow +\infty]{} u_\infty \text{ weakly in } H^2(M),$$

and

$$u(t_n) \xrightarrow[n \rightarrow +\infty]{} u_\infty \text{ in } C^\alpha(M), \quad \alpha \in (0, 1). \quad (4.24)$$

It is easy to check that  $u_\infty$  is a weak solution of

$$\Delta u_\infty - \frac{\rho}{|M|} + \frac{\rho f e^{u_\infty}}{\int_M f e^{u_\infty} dV} = 0,$$

and, by bootstrap regularity arguments, we have  $u_\infty \in C^\infty(M)$ . We claim that  $\|u(t_n) - u_\infty\|_{H^2(M)} \xrightarrow{n \rightarrow +\infty} 0$  strongly in  $H^2(M)$ . To prove the claim, in

view of (4.24), it is sufficient to show that  $\int_M (\Delta u(t_n) - \Delta u_\infty)^2 dV \xrightarrow{n \rightarrow +\infty} 0$ .

Using Hölder's inequality, we have

$$\begin{aligned} \int_M (\Delta u(t_n) - \Delta u_\infty)^2 dV &= \int_M \left( \rho \left( \frac{f e^{u_\infty}}{\int_M f e^{u_\infty} dV} - \frac{f e^{u(t_n)}}{\int_M f e^{u(t_n)} dV} \right) + \frac{\partial e^{u(t_n)}}{\partial t} \right)^2 dV \\ &\leq C \int_M \left( \frac{e^{u_\infty}}{\int_M f e^{u_\infty} dV} - \frac{e^{u(t_n)}}{\int_M f e^{u(t_n)} dV} \right)^2 dV \\ &\quad + C \int_M \left( \frac{\partial e^{u(t_n)}}{\partial t} \right)^2 dV. \end{aligned} \quad (4.25)$$

Since  $|u(t)| \leq C$ ,  $\forall t \geq 0$ , we deduce from (4.23) that

$$\int_M \left( \frac{\partial e^{u(t_n)}}{\partial t} \right)^2 dV \xrightarrow{n \rightarrow +\infty} 0. \quad (4.26)$$

Let's denote  $\beta = \frac{\int_M f e^{u(t_n)} dV}{\int_M f e^{u_\infty} dV}$ . Using the estimate  $|e^x - 1| \leq |x| e^x$  and Hölder's inequality, we have

$$\begin{aligned} &\int_M \left( \frac{e^{u_\infty}}{\int_M f e^{u_\infty} dV} - \frac{e^{u(t_n)}}{\int_M f e^{u(t_n)} dV} \right)^2 dV \\ &= \frac{1}{\left( \int_M f e^{u(t_n)} dV \right)^2} \int_M e^{2u(t_n)} \left| e^{u_\infty - u(t_n) + \ln \beta} - 1 \right|^2 dV \\ &\leq C \int_M e^{2u(t_n)} |u_\infty - u(t_n) + \ln \beta|^2 e^{2(u_\infty - u(t_n) + \ln \beta)} dV \\ &\leq C \left( \int_M e^{8u(t_n)} dV \right)^{\frac{1}{4}} \left( \int_M |u_\infty - u(t_n) + \ln \beta|^2 dV \right)^{\frac{1}{2}} \\ &\quad \times \left( \int_M e^{8(u_\infty - u(t_n) + \ln \beta)} dV \right)^{\frac{1}{4}}. \end{aligned}$$

From the previous inequality, (4.24) and since  $\int_M f e^{u(t_n)} dV \xrightarrow{n \rightarrow +\infty} \int_M f e^{u_\infty} dV$ , we obtain that

$$\int_M \left( \frac{e^{u_\infty}}{\int_M f e^{u_\infty} dV} - \frac{e^{u(t_n)}}{\int_M f e^{u(t_n)} dV} \right)^2 dV \xrightarrow{n \rightarrow +\infty} 0. \quad (4.27)$$

Therefore, from (4.25), (4.26) and (4.27), we deduce that

$$\int_M (\Delta u(t_n) - \Delta u_\infty)^2 dV \xrightarrow{n \rightarrow +\infty} 0.$$

Thus, we proved that  $\|u(t_n) - u_\infty\|_{H^2(M)} \xrightarrow{n \rightarrow +\infty} 0$  strongly in  $H^2(M)$ . Finally, since the flow (1.3) is a gradient flow and  $E_f$  is real analytic, by using a general result from Simon [18], we derive that

$$\|u(t) - u_\infty\| \xrightarrow{t \rightarrow +\infty} 0.$$

This concludes the proof of Theorem 1.2.

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